

AN
EXPLANATION
OF THE
AFFIRMATIVE and NEGATIVE
SIGNS
IN
ALGEBRA.

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EXPLANATION

OF THE

ALPHABETIC AND NUMERATIVE

SYMBOLS

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ALPHABETIC

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P R E F A C E.

THIS Paper took its rise from the following Quære. “ — $a a$ is allowed on all hands to be possible quantity, why then should its root be impossible?” This necessarily leads to a philosophical disquisition on the distinct natures of real and imaginary quantity; and it is presumed that — $a a$, in the algebraical meaning of the expression, is demonstrated to be itself in the number of imaginary or impossible quantities, of which there are innumerable gradations; and that one root, whether it be square, cube, biquadrate, &c. of every real quantity is real, whilst every root of every imaginary quantity is imaginary. As it has been objected to Mathematicians, that they have not understood the principles of the art which they have professed

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fessed to treat of, it is of some consequence to explain what seems to carry the air of a mystery in it; as it will serve, if the following observations on the subject are just, to correct some inaccurate expressions to be met with in books of Algebra; tho' it is acknowledged the practice will not be affected by it.

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EXPLANATION, &c.

ALGEBRA is the art of reasoning on quantity and its qualities, by means of certain signs, which instead of ordinary language, express the operations of the mind. For the right understanding these signs therefore, nothing more is requisite, than a due attention to that ordinary language, the use of which they are intended to supply.

The quality of real quantity is the mode of its existence.

All quantity may be considered as existing either *absolutely*, or in a *certain mode*; and as the mind has a power of contemplating ei-

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ther of these existences, and of reasoning concerning them, the signs of Algebra, which are substituted in the place of ordinary language, may certainly be made use of to express this reasoning in either view.

First, Suppose that a velocity, without regard to the mode of its direction, is expressed by the quantity $+a$; this not only expresses its quantity, but also expresses it as existing; and it is this existence of (a) as a measure of velocity, that is here understood by its *absolute existence, as being such a measure*. Suppose this velocity to be destroyed; this, in common language, is, (a) is destroyed, and in the language of Algebra is $-a$. It is the same individual object which is concerned in the two expressions $+a$, and $-a$; but considered in the contrary circumstances of being and not being; and the expression $-a$, when set in apposition with $+a$, is merely a declaration that the affirmed existence of a is now destroyed; and $-a$ itself, when considered as quantity distinct from $+a$, is the quantity a declared

clared to be destroyed. It has indeed a farther meaning, but it is such as is not necessary to be considered in this place. If it should be said, that a velocity of four times the quantity of a is in being, the algebraic expression for this is 4 into a ; and if a velocity of four times the quantity of a is destroyed, the expression for this is 4 into $-a$; i. e. 4 into a ; and as the destruction of a is declared; —, the sign of that destruction, is prefixed to the product. If it should be said, that the destruction of four times a , which was before declared, is all a mistake; the expression for this is -4 into $-a$; i. e. 4 into a ; and, as *the destruction is denied*, +, the mark of existence, is prefixed to the product.

From the premises it is concluded; first, That in this system of being + affirms the existence of those quantities to which it is prefixed, and — denies their existence. Secondly, when a quantity to which + is prefixed, is multiplied by another quantity to which the same sign is prefixed, that the existence of the product

duct is then affirmed; but when it is multiplied by a quantity to which — is prefixed, that the existence of the product is denied. Thirdly, that when two quantities, which have each the sign — prefixed, are multiplied together, the existence of the product is then affirmed. Fourthly, that — a being the quantity a denied to exist; if n be any odd number, and the n root of a be taken, and that root be b ; then b denied to exist is the n root of a denied to exist; but if n be an even number, the n root of — a is not then assignable in the terms of quantity, either intirely affirmative, or intirely negative. Fifthly, that — a , as a measure of a velocity, is quantity without real existence. In another view, it has certainly such existence, but this is not attended to at present. The mind for its convenience gives it an imaginary reality in this system; and whilst it is considered as what in truth it is, it may, like any other imaginary quantity, be so far looked upon as possible.

Secondly, Let now $+ a$ express the quantity

tity of a velocity in some certain mode of direction $\overset{C}{\text{---}} \overset{A}{\text{---}} \overset{B}{\text{---}}$ as AB , affirming its existence in this direction. This existence of a as a measure of velocity, is what is here understood by its *modal existence*, *as being such a measure*. Suppose this velocity to be destroyed; the algebraic expression for this is $-a$, the sign prefixed denying its existence in the same direction. Hence $-a$, considered as real in this system, is imaginary quantity; and for the same reason as $-a$, in the preceding system; and neither the one nor the other expresses quantity of velocity in a contrary direction, the absolute being of (a) as a measure of real velocity, being in both cases denied.

Thirdly, Let now the lost velocity be restored, but in the contrary direction AC . The expression for this is $-a$; (any real velocity of the same quantity, in the direction AB , being still supposed to be expressed by $+a$) and now the very same quantity, which is denied to exist as a measure of velocity in the
system

system AB , has a real existence, as being such a measure, in another modal system AC ; for now the signs reciprocate. In the system AB $+$ affirms and $-$ denies; in the system AC , $-$ affirms and $+$ denies the existence of quantity. Here $- a$ into $- a$, and $+ a$ into $+ a$, make each $- aa$, affirming the existence of quantity; and $+ a$ into $- a$ is $+ aa$, denying its existence in the system AC , but affirming it in the system AB . But tho' $-$ thus denies the existence of quantity in one system, and affirms it in another, yet excepting this complex declaration, which is chiefly to be seen in the construction of a locus, or the answer to a problem, it is only the former of the two meanings which is attended to in an algebraic process. Hence a negative quantity with its several roots, which when viewed in one system are all imaginary, are all real when viewed in another system. As the negation of quantity in one system is intirely consistent with the affirmation of its existence in another, and is correspondent with it, the facility in extracting those roots of negative quantities,

tities, which are denominated by odd numbers, is easily accounted for. In the system AB , the cube root of $-a^3$ is $-a$, where $-$ denies existence; and in the system AC , the cube root of $-a^3$ is $-a$, where the same sign affirms existence. Here the laws of the two systems meet in correspondence, producing consistent effects necessarily connected with each other. This correspondence is likewise easily exemplified in the instance of the cubical parabola. A similar correspondence is observable, when the even roots of positive quantities are extracted. Let the square of an ordinate in any circle be $+aa$, the side of this square, in the algebraic expression, is $+a$ or $-a$, i. e. (a) denied to exist in the same direction with $+a$, which negation is necessarily connected with its real existence in a contrary direction; and here the sign is changed from negative to affirmative, as is easy to be seen. In the system AC , $-aa$ is real, and its root $-a$ is real; for one root of every real quantity is and must be possible. By real quantity is here meant *such as is conceived to have*

have real existence, and is employed to express objects in their real existing state. These observations will shew in what sense we are to understand such expressions as this, "that — a is real, but its root imaginary:" two opposite systems are here plainly alluded to.

In the series $b + c - a$, the first quantity having no sign prefixed, has $+$ understood before it to mark its nature; and the same sign prefix'd to the second, both marks its nature, and directs its application. The negative sign prefix'd to the third, marks the nature of it negatively, and directs to subtract an affirmative quantity; but if the whole expression — a be considered as quantity, it has $+$ understood before it, to give it an imaginary existence in the same system with $+ a$, and to connect it with other parts of the system. By the imaginary existence of quantity as opposed to real, is meant *such as, being no existence in nature, is conceived against nature, for some particular use.*

As $+ - a$ expresses possible quantity in the sense of its being imaginary, so $- - a$ expresses impossible quantity in the same sense, or, what is the same in effect, it expresses possible real quantity. As the square root of $- a$ is neither intirely affirmative, nor intirely negative, it must have the absurdity of partaking at once of both these natures in the same system. As Algebra has no single sign for the purpose, let the arbitrary mark $\neq$ express this compound meaning; then $\neq a$ is the root of $- a$, & $+$ understood before it gives it an imaginary reality. In the expression $- \neq a$, the force of the negative sign changes all that is negative in $\neq a$ into affirmative; and at the same time all that is affirmative into negative; by which means the quantity preserves a mixt nature, and is impossible, tho' it is opposite to $+\neq a$. This observation extends to the even roots of negative quantities ad infinitum. As $- a$ is imagined to be real, for the purpose of destroying $+\neq a$, so $\neq a$ is imagined to be real, partly to be destroyed by its opposite, and

B

partly,

partly, after it is multiplied by quantity of a mixt nature, to make a product of such kind, as, being added to real quantity, may either increase or diminish it, according as the quantity to be added is real or imaginary. There is this difference between the expressions $-a$ & $\sqrt{-a}$; the former is the name of quantity both real and imaginary; the latter of imaginary only.

Thus far two sorts of imaginary quantity have been considered: but there are innumerable sorts which deviate more and more from real quantity, as appears by extracting the square root of $-a$ in continuance ad infinitum. For in the several expressions $-a$, $\sqrt{-a}$, $\sqrt[4]{-a}$, $\sqrt[8]{-a}$, &c. the greater the index of the root to be extracted is, the more the root, when supposed to be extracted, is removed from reality. For when the bi-quadrature root of $-a$ is extracted, this root requires more continued multiplications of itself into unit to produce real quantity, than the square root of $-a$; and this again requires

quires more continued multiplications to produce real quantity than $-a$ itself. Hence the several expressions $-a$, $\sqrt[2]{-a}$, $\sqrt[4]{-a}$, $\sqrt[8]{-a}$, &c. may not improperly be said to be of the first, second, third, &c. sort, order, or degree of imaginary quantity.

When $-a$ expresses real quantity, the sign prefixed is affirmative in its meaning, and shews the quantity to be of an opposite quality from $+a$. For tho' the sum of these quantities, thus considered, is really $2a$, part being of one quality, and part of another, yet the different signs give notice, *that they are opposite to each other, in some intended estimation*, and that the real quantity $-a$ is not to be added to, but subtracted from the real quantity $+a$, to effect the intended purpose. If the real quantity $+a$ be greater than the real quantity $-b$, which is proposed to be subtracted from it, the process is practicable by the established rule of nature; but if it is less, the process is impracticable by that rule;

but the mind, for its convenience, imagines such subtraction possible, expressing the imaginary remainder, in this case, by a quantity under no circumstance of real existence, but which the mind still considers as existing: it is therefore impossible. For quantity may be truly said to be impossible, when something impossible is supposed to be involved in the expression of it. As it is impossible, that any quantity should have real existence, which is divested of its reality; so $-a$, which is an expression of quantity supposed to be thus divested, considered as real, is impossible. As it is impossible to find a real quantity, which multiplied into itself shall make $-aa$; so $\neq a$, which multiplied into itself, is supposed to make such a product, is impossible real quantity.

In this doctrine, when it is said that $-a$ into $-b$, or that $+\sqrt{-aa}$ into $-\sqrt{-bb}$, makes a product or rectangle, which is expressed by $+ab$, it is not meant, that any rectangled parallelogram can have real existence

ence without real sides to contain it; but that if one of the real sides of a real rectangle, the contents of which are $+ab$, be express'd by the imaginary quantity $-a$; an adjacent side of the same rectangle must be express'd by $-b$; or if one of the real sides, &c. be express'd by $+\sqrt{-aa}$, an adjacent side must be express'd by $-\sqrt{-bb}$.

The following example is applicable to the first of the three cases proposed. The velocities of three bodies moving in various directions, none of which are in any wise contrary to each other, are in continued proportion: The sum of the first and second is 10, and the difference of the second and third is 24. What are the velocities? The answer is 2, 8, 32; or 25, — 15, 9. The former numbers only are the true answer, the latter containing an imaginary quantity, which is indeed necessarily connected with real quantity in an abstracted sense (for such is — 15 at present when it is real); but has no connection with
real

real quantity as a measure of real velocity.—
 If all the bodies be supposed to move in one and the same direction, the example thus altered will suit the second case; and the answer will be the same. Suppose now, that two of the bodies are moving in the direction AB , and the third in the direction AC ; if the two velocities in the direction AB be called $+x$ & $+y$, that in the direction AC may be called $-z$, where the sign is affirmative; and the example is made to suit the third case. To accommodate this to algebraical purposes; as that which has a positive existence in the direction AC may be supposed to have a negative existence in the direction AB ; in this sense $-z$ is negative, and the three velocities are all estimated in one direction. The proper answer to the question thus considered is 25, -15 , 9; where the imaginary quantity -15 is immediately connected with real quantity, as a measure of velocity, in a different direction,

If all the money which A has be called $+a$,
 and

and all the money he *owes* be called $-b$, both these quantities are alike real, because the signs in both cases are affirmative; and, in ascertaining what A is worth, they are also of opposite qualities to each other, because the real quantity $-b$ must be subtracted from the real quantity $+a$ for this purpose. But if $-b$, instead of being considered as a real sum of money which A *owes*, be understood to mean a sum of money which A *has*, over and above the real quantity $+a$ (in which light it is viewed in Algebra); it then signifies a quantity which, having no reality in nature, is imagined to be real for a due estimation of A 's property.

Let two equal forces, in contrary directions, be supposed to act at once upon a body at rest, in the point A , in the line CB . If the force in the direction AB be considered as in real existence, it may be expressed by the quantity $+a$. The body by means of this force only would move in the direction AB , and no real additional quantity of force in the
same

same direction could prevent its motion; consequently any additional quantity of force, supposed to be exerted in this direction sufficient to prevent all motion, must be imaginary or impossible. If the real force in the direction AC be called $-a$, the sign will be then affirmative; but if this force be considered as abstracted from reality, and estimated as to its effects in the direction AB , then $-a$ is negative, and expresses an imaginary quantity of force acting in the direction AB , which, in respect of $+a$, is quantity less than nothing, and therefore impossible. When two real quantities are such as are opposite to each other in their effects, it is usual, in Algebra, to substitute an imaginary quantity in the place of one of them; and when the process is finished (if the nature of the proposed question will admit of it), to consider the answer, if negative in its estimation, as real quantity.



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